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On the Boltzmann–Grad Limit for the Broadwell Model of the Boltzmann Equation

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An example is given of a model dynamics for which the Broadwell model of the Boltzmann equation seems to appear in the formal stage of the Boltzmann–Grad limit, but actually does not.

KEY WORDS: Model dynamics; Grad limit; derivation of the Boltzmann equation.

1. INTRODUCTION

In this paper we introduce a dynamical system that seemingly corresponds to the Broadwell model of the Boltzmann equation and discuss the Grad (or Boltzmann–Grad) limit for it. For the hard-sphere model Lanford[(7)] derived the Boltzmann equation from the BBGKY hierarchy for short times by taking the Grad limit. For the present model an analogy is observed at the formal level. The BBGKY hierarchy (which may be obtained by a formal computation) is valid and its first (or rather last) equation is reduced to the Broadwell model by formally passing to the Grad limit. In the actual limit, however, the solution of the Boltzmann equation does not come out.

The Broadwell model of the Boltzmann equation in R^2 is the following nonlinear equation:

$$\frac{\partial}{\partial t} u(t, x) + v \cdot \frac{\partial}{\partial q} u(t, x) = a[u(t, q, iv)\, u(t, q, -iv) - u(t, q, v)\, u(t, q, -v)] \tag{1.1}$$

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where a is a positive constant, $x=(q, v)$, and $q \in R^2$, while the velocity v takes only four possible values from the velocity set

$$S := \{(1,0), (0,1), (-1,0), (0,-1)\} \tag{1.2}$$

and i denotes the rotation operator, which rotates a two-dimensional vector by $\pi/2$ around the origin counterclockwise.[2] An integrated version of (1.1) is

$$u(t) = U_1^0(t) f + a \int_0^t U_1^0(t-s)[u(s)^{2\otimes}]^0 \, ds \tag{1.1'}$$

where

$$[u(t)^{2\otimes}]^0(x) = u(t, q, iv)\, u(t, q, -iv) - u(t, q, v)\, u(t, q, -v)$$

and $U_1^0(t)$ is a group of operators for one-particle free motion: $U_1^0(t) f(x) = f(q - tv, v)$. For each bounded $f \in C(\Omega_1^0)$ $(\Omega_1^0 := R^2 \times S)$ there is a unique bounded solution $u(t) \in C(\Omega_1^0)$ to Eq. (1.1′) on an interval $0 \leqslant t < (2a \|f\|_\infty)^{-1}$. (The existence results of global solutions are obtained under smallness conditions of initial values (e.g., ref. 6) or in the case where the initial values are trivial along one space coordinate (if the dimension is two) (e.g., ref. 8)).

Let us briefly describe our dynamical system, which is supposed to correspond to (1.1). Let Λ be a closed square in R^2 whose four vertices are $(\pm 1, 0)$, $(0, \pm 1)$. Given $\varepsilon > 0$, a particle is a square in R^2 that is a translation of the shrunken square $\frac{1}{2}\varepsilon\Lambda$. Each particle moves with a constant speed v from the velocity set (1.2) between successive collisions. Collisions between two particles take place as illustrated in Fig. 1. Let us consider a dynamical system of n such particles. It is shown that the triplet or higher order collisions may be disregarded, so that the flow of the dynamical system is determined for all times for almost all initial phases. Suppose that the initial phase is randomly distributed according to a probability measure

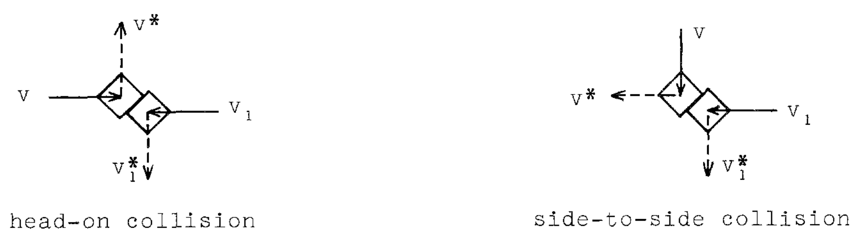

Fig. 1. In our dynamics there are only two (essentially distinct) types of collisions as diagrammed, where v and v_1 (v^* and v_1^*) denote the precollisional (postcollisional) velocities.

that has a symmetric density (with respect to the "Lebesgue" measure), denoted by f_n. Let $u_{n|m}(t)$ be the density function for the phase of the first m particles at time t ($m \leqslant n$). Then we can derive a chain of equations for $\{u_{n|m}\}_{m=1}^{n}$, called the BBGKY hierarchy, the first of which reads

$$u_{n|1}(t) = U_1^0(t) f_{n|1} + 4\varepsilon(n-1) \int_0^t U_1^0(t-s)[u_{n|2}(s)]^{(\varepsilon)}\, ds \quad (1.3\text{a})$$

$$[u_{n|2}(t)]^{(\varepsilon)}(x) = \frac{1}{2\sqrt{2}} \int_{\substack{v \cdot l > 0 \\ v_1 \cdot l < 0}} [u_{n|2}(t, q, v^*; q - \varepsilon l, v_1^*) - u_{n|2}(t, q, v; q + \varepsilon l, v_1)]\, dl\, dv_1 \quad (1.3\text{b})$$

where dl is a line element of the boundary $\partial\Lambda$ and dv_1 is a discrete measure that assigns unit mass to each velocity. Let us put $\varepsilon = 1/n$ and consider the limit as $n \to \infty$ (the Boltzmann–Grad limit), assuming that the family $\{f_n\}$ is chaotic with the limiting one-particle density f. Then, if $u_{n|2}(t, x_1, x_2)$ is assumed to split into a product $u(t, x_1)\, u(t, x_2)$ in the limit, the Boltzmann equation (1.1′) with $a = 4$ would result from (1.3), having $u(t) = \lim u_{n|1}(t)$ as its unique solution. This turns out to be false. The fact of the matter is that $u_{n|m}(t)$, being convergent, is factorized in the limit for m particles of general configurations, but not for those of such special ones as appearing in (1.3b), and accordingly the common factor $\lim u_{n|1}(t)$, which is shown to be continuous, does not solve (1.1′) (see Section 2 for a precise statement).

The plausibility of the specious hypothesis that $u_{n|2}(t)$ in (1.3) be factorized in the limit may be accounted for, as in the classical case, as follows. First we observe the apparent fact that the two-particle configurations appearing in the integrand on the right-hand side of (1.3b) are those of incoming collisions, i.e., of two particles that are about to collide. We then note that, on one hand, we may virtually neglect all the side-to-side collisions (cf. Fig. 1), since their effect on the correlation functions disappears in the limit, and, on the other hand, the head-on collisions are rarely repeated between the same pair of particles—this is not the case for the side-to-side collisions—and there seems to be no reason for two particles that are about to make a head-on collision with each other to have interacted in the past (but after time zero) directly or indirectly through head-on collisions (see Fig. 2). If so, the correlation function for such particles (like that for those in general configurations) would inherit the chaos property of the initial distribution: thus, the hypothetical factorization of $u_{n|2}(t)$ in the limit.

What is wrong with the reasoning above? If we interpret the interactions as collisions between the two particles or their "ancestoral" particles

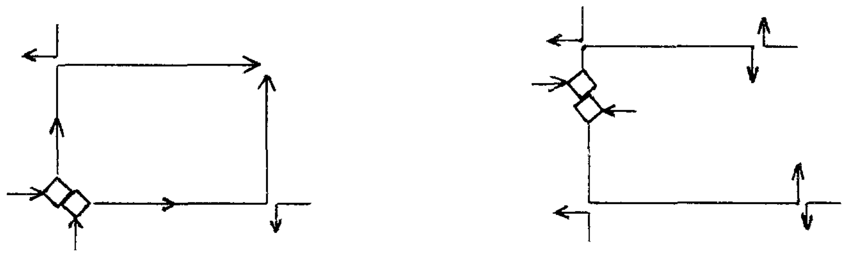

Fig. 2. (Left) Side-to-side collisions are repeated between the same pair of particles with a good probability; (right) head-on collisions are not.

(i.e., their being traced back to a common "progenitor" particle), then it is negligible (as shown in the Appendix), but its negligibility does not imply the chaos property in question. The fact is that what causes correlations between our two particles is not these negligible collisions, but an interaction of a remoter kind (see discussion in the last part of Section 2).

For the mathematical proof of our claim, given in Section 3, we employ the series expansion of $u_{n|1}(t)$ obtained from the BBGKY hierarchy, which will display the mechanism that makes the Boltzmann equation (1.1′) fail to emerge from our dynamics: in the addition-backward flow evolution that determines the correlation function via the series expansion there persistently take place collisions (other than those caused by the particle additions) throughout in our passing to the Grad limit for the present model—this is not the case for the hard sphere model.

Grad[4] afforded an excellent insight into the problem of deriving the Boltzmann equation from the Liouville equation, in which he seemed to anticipate some crucial points in Lanford's derivation. This may be paraphrased that the latter, being concrete and mathematically presented, makes transparent the (corresponding) parts of what Grad advanced in a general setting without mathematical precision. The analysis of the present model, for which the Boltzmann equation does not appear in the Grad limit, would serve as another illustration of Grad's remark that a very small part of the n-particle phases controls the formation of the one-particle distribution; in our model that very part badly behaves, which is not apparent unless the situation is carefully examined.

Most of the results of this paper were announced in ref. 10.

2. THE MODEL DYNAMICS, THE BOLTZMANN–GRAD LIMIT, AND THE RESULT

This section is divided into four parts. The model dynamics is described in the first part. The corresponding BBGKY hierarchy, which is introduced in the second part, would seem to be reduced to the Boltzmann hierarchy that corresponds to Eq. (1.1) if one formally passes to the Boltzmann–Grad limit. This is denied by Theorem 1 of the third part, where the actual limit is compared with the solution to Eq. (1.1). Also in the third part we state another theorem, Theorem 2, that expresses in what sense $\{u_n(t)\}$ is chaotic. Some intuitive reasoning that may account for the main claim in Theorems 1 and 2 is advanced in the fourth part. The proofs of Theorems 1 and 2 are given in Sections 3 and 4, respectively. In the first two parts below we state several facts without proof. For these see ref. 11 (also see refs. 1, 5, and 9).

2.1. The Description of the Model

Let Λ be a square (a closed domain) in R^2 whose four vertices are $(\pm 1, 0)$, $(0, \pm 1)$. A particle is a square in R^2 that is a translation of the shrunken square $\frac{1}{2}\varepsilon\Lambda$ $(\varepsilon > 0)$; hence the length of its diagonal is ε. The position of a particle is represented by a point of intersection of its diagonals. Thus, a particle located at q is a square whose vertices are $q + (\pm\varepsilon/2, 0)$, $q + (0, \pm\varepsilon/2)$. Each particle moves with a constant speed $v \in S := \{(\pm 1, 0),\ (0, \pm)\}$ between successive collisions. A collision between two particles takes place when they properly touch each other with their sides, i.e., they come into positions q and q_1 such that

$$l := \frac{1}{\varepsilon}(q_1 - q) \in \partial\Lambda \backslash \{(\pm 1, 0), (0, \pm 1)\}$$

For the extremal case $l \in \{(\pm 1, 0),\ (0, \pm 1)\}$ the whole system is stopped and sent to the extra state ∂ at the moment of touching. Let v and v_1 be the velocities of two particles before the collision. For a possible collision

$$l \cdot v > 0 \qquad \text{and} \qquad l \cdot v_1 < 0$$

Velocities v^* and v_1^* after collision are defined by

$$v^* = \sigma i v, \qquad v_1^* = \sigma i v_1 \qquad \text{if} \quad v \cdot v_1 = -1 \tag{2.1}$$

$$v^* = v_1, \qquad v_1^* = v \qquad \text{if} \quad v \cdot v_1 = 0 \tag{2.2}$$

where $\sigma = 1$ or -1 according as $(iv) \cdot l < 0$ or $(iv) \cdot l > 0$, so that

$$l \cdot v^* < 0 \qquad \text{and} \qquad l \cdot v_1^* > 0$$

(see Fig. 1). We shall call collisions of the type (2.1) [resp. (2.2)] head-on [resp. side-to-side]. The multiple (i.e., triple or higher order) collision is undefined and when the system comes into a configuration of multiple touching it is sent to ∂. This virtually defines the dynamical system of n particles whose phase space is

$$\Omega_n = \Omega_n^{(\varepsilon)} := \{\mathbf{x} = (x_1;...; x_n): \varepsilon^{-1}(q_i - q_j) \notin \Lambda \backslash \partial\Lambda \text{ if } i \neq j\}$$

where $x_i = (q_i, v_i)$, $q_i \in R^2$, and $v_i \in S$. The boundary of Ω_n is given by

$$\partial\Omega_n = \{\mathbf{x} = (x_1;...; x_n) \in \Omega_n : \varepsilon^{-1}(q_i - q_j) \in \partial\Lambda \text{ for some } i \neq j\}$$

For the sake of convenience we identify with ∂ every formal configuration in which at least two particles occupy a region in common and put $f(\partial) = 0$ for any function f defined on Ω_n, understanding the extra point ∂ to be added to Ω_n as an isolated point. Let $T_t\mathbf{x}$, $t \in R$, be the left continuous version of the trajectory in $\Omega_n \cup \{\partial\}$ traced by the system starting at $\mathbf{x} \in \partial\Omega_n$ at time zero (the left continuity is asserted as long as the system is in Ω_n).

Let dv stand for a discrete measure on the velocity space S which charges each point with unit mass and put $d\mathbf{x} = dx_1\, dx_2 \cdots dx_n$ and $dx_i = dq_i\, dv_i$. Then the n-particle phases (i.e., the configurations of positions and velocities of n particles) that eventually (in past or future) lead to multiple or corner-to-corner collisions form a $d\mathbf{x}$-null set, and the flow T_t preserves the measure $d\mathbf{x}$. For the proof of the latter statement as well as a manipulation of the collision integral the following relation is crucial: for every bounded, measurable function F on $S^4 \times \partial\Lambda$

$$\int_{\substack{v \cdot l > 0 \\ v_1 \cdot l < 0}} F(v, v_1; v^*, v_1^*, l)\, dv\, dv_1\, dl$$

$$= \int_{\substack{v \cdot l > 0 \\ v_1 \cdot l < 0}} F(v^*, v_1^*; v, v_1, -l)\, dv\, dv_1\, dl \tag{2.3}$$

where the integrals extend over all points of $S^2 \times \partial\Lambda$ that satisfy the indicated constraint.

We shall call $d\mathbf{x}$ the "Lebesgue" measure (on Ω_n). The measure on $\partial\Omega_n$ naturally induced from it we also call "Lebesgue" measure. The measure-theoretic terms such as "a.e." or "Lebesgue null" will refer to the corresponding notions with respect to these "Lebesgue" measures. When we want to elucidate the dependence on n or ε of T_t, we shall write $T_t^{(n)}$, $T_t^{(\varepsilon)}$, or $T_t^{(n,\varepsilon)}$.

2.2. The BBGKY Hierarchy

Let $f_n = f_n(\mathbf{x})$ be a bounded and integrable Borel function on Ω_n that is symmetric, i.e., invariant under any permutation of components $x_1,..., x_n$. The image measure of $f_n(\mathbf{x})\, d\mathbf{x}$ under T_t has a density given by

$$u_n(t, \mathbf{x}) = f_n(T_{-t}\mathbf{x})$$

We denote by $u_{n|m}(t)$ its marginal density for $x_1,..., x_m$:

$$u_{n|m}(t, x_1;...; x_m) = \int u_n(t, x_1;...; x_m; x_{m+1};...;x_n)\, dx_{m+1} \cdots dx_n$$

$(m = 1,..., n-1)$ and put $u_{n|n}(t) = u_n(t)$. We shall assume

$$f_n \text{ is continuous at a.a. points of } \Omega_n \tag{2.4}$$

i.e., there exists a Lebesgue null set A of Ω_n such that f_n is continuous at each point of $\Omega_n \backslash A$. [Instead of or in addition to (2.4) we may assume the continuity along trajectories as in refs. 5, 7, or 9 without giving rise to any essential change in what follows.] Under the condition (2.4) the evolution of $u_{n|m}(t)$ is described by the following system of equations: for all $t \in R$ and for $1 \leqslant m < n$

$$u_{n|m}(t) = U_m(t) f_{n|m} + \varepsilon(n-m) \int_0^t U_m(t-s) K_{m,m+1} u_{n|m+1}(s)\, ds \quad \text{a.e. on } \Omega_n \tag{2.5}$$

where $U_m(t)\, g(\mathbf{x}) = g(T_{-t}^{(m)}\mathbf{x})$, $\mathbf{x} \in \Omega_m$ (the evolution operator for the m-particle system) and

$$\begin{aligned} &K_{m,m+1}\, g(x_1;...; x_m) \\ &\quad = \sqrt{2} \sum_{k=1}^{m} \int_{\substack{v \cdot l > 0 \\ v_k \cdot l < 0}} \{ g(...; \underbrace{q_k, v_k^*}_{k\text{th entry}};...; \underbrace{q_k - \varepsilon l, v^*}_{(m+1)\text{th entry}}) - g(...; \underbrace{q_k + \varepsilon l, v}_{(m+1)\text{th entry}}) \}\, dl\, dv \end{aligned}$$

The operator $K_{m,m+1}$ may act at least on a bounded function g on Ω_{m+1} which is continuous at a.a. boundary points of Ω_{m+1}, i.e., there exists a Lebesgue null set B of $\partial\Omega_{m+1}$ such that g, as a function on Ω_{m+1}, is continuous at each $\mathbf{x} \in \partial\Omega_{m+1} \backslash B$. It is shown that

$$\begin{aligned} &\text{if } A \text{ is a Lebesgue null set of } \Omega_n, \\ &\qquad \text{then } T_t\mathbf{x} \notin A \text{ for a.a. } (t, \mathbf{x}) \in R \times \partial\Omega_n \end{aligned} \tag{2.6}$$

and, by applying (2.6), that under (2.4) the continuity condition for g mentioned above is satisfied by $u_{n|m+1}(t)$ for a.a. t. The chain of equations (2.5) is the *BBGKY hierarchy* for the present model. On the formal level it is deduced by having the whole interaction effect approximated with the sum of pairwise ones (cf. Section 1 of ref. 11), or, as in ref. 3, by applying the Gauss divergence theorem, the justification of which, however, might need some nontrivial analysis. In any case we can prove (2.5) under (2.4). One can iterate the relation (2.5) to obtain

$$u_{n|m}(t, \mathbf{x}) = \sum_{k=0}^{n-m} (n-m)_k \varepsilon^k (\mathscr{U}K)^k \{U_{m+k}(\cdot) f_{n|m+k}\}(t, \mathbf{x}) \qquad \text{a.e. on } \Omega_m \tag{2.7}$$

for all $t \geq 0$, where $(n)_0 = 1$, $(n)_k = n(n-1)\cdots(n-k+1)$,

$$\mathscr{U}_m f(t, \mathbf{x}) = \int_0^t U_m(t-s) f(s, \cdot)(\mathbf{x})\, ds, \qquad f = f(s, \mathbf{x})$$

$$(\mathscr{U}K)^k = \mathscr{U}_m K_{m,m+1} \cdots \mathscr{U}_{m+k-1,m+k}$$

2.3. Statement of the Result

For a bounded, continuous f the (unique) local solution of Eq. (1.1′) is given by the infinite series

$$u(t, x) = \sum_{k=0}^{\infty} (\mathscr{U}^0 K^0)^k (U_{k+1}^0(\cdot) f^{(k+1)\otimes})(t, x) \tag{2.8}$$

which converges uniformly in $0 \leq t \leq t_0$, $x = (q, v)$ if $t_0 < (8\,\|f\|_\infty)^{-1}$. Here U_m^0 and $\mathscr{U}_m^0$ are defined as U_m and $\mathscr{U}_m$ but with the flow T_t^0 of the free motion of m particles in place of $T_t^{(\varepsilon)}$, and $K_{m,m+1}^0$ is as $K_{m,m+1}$ by putting $\varepsilon = 0$ in the integral that defines the latter; $\otimes$ denotes the outer product, e.g., $f^{2\otimes}(x_1; x_2) = f(x_1) f(x_2)$. The phase space of m point particles is denoted by Ω_m^0. Now let us take the Grad limit, which is the limit or the process of passing to the limit such that $n \to \infty$ and $n\varepsilon \to 1$ (in Theorems 1 and 2 below we shall put $\varepsilon = 1/n$ for the sake of simplicity). Since σ in (2.3) takes values 1 or -1 with equal weight in (2.7) and the effect of side-to-side collisions on the correlation functions disappears in the Grad limit, the series expansion (2.7) with $m = 1$ seems to lead to (2.8), or on a more formal level the relation (1.3) seems to be reduced to (1.1′) if $u_{n|2}(t)$ is factorized in the limit. This is not true. The function $u_{n|m}(t, \mathbf{x})$ converges and, except for a very small set of $\mathbf{x}$, is factorized in the Boltzmann–Grad limit if $f_{n|m}$ does and is, but the common factor $\lim u_{n|1}(t)$ does not solve

(1.1′). The precise statement are given in the following theorems, where we put

$$d_m = \{\mathbf{x} \in \Omega_m^0 : q_i = q_j \text{ for at least one pair } i \neq j\}$$

Theorem 1. Let f_n be a symmetric Borel function on $\Omega_n^{(1/n)}$ satisfying (2.4), $n = 1, 2, \ldots$. Assume that there are positive constants C and M and a continuous function f on Ω_1^0 such that, for $m = 1, 2, \ldots$,

$$\|f_{n|m}\|_\infty \leqslant CM^m \qquad \text{for} \quad m \leqslant n$$

and

$$f_{m|n} \to f^{m\otimes}(n \to \infty) \qquad \text{uniformly on each compact set of } \Omega_m^0 \backslash d_m$$

Then (i) for every $m = 1, 2, \ldots$, $0 \leqslant t < (16M)^{-1}$, the right-hand side of the series expansion (2.7) with $\varepsilon = 1/n$ converges to a limit function, denoted by $u^{(m)}(t, \mathbf{x})$, as $n \to \infty$ for all $\mathbf{x} \in \Omega_m^0$ and the convergence is uniform on each compact set of the (open) set $J_m(t) := \{\mathbf{x} \in \Omega_m^0 : T_{-s}^0 \mathbf{x} \notin d_m \text{ for all } 0 \leqslant s \leqslant t\}$; hence, for each compact set $K \subset J_m(t)$

$$\operatorname*{ess\,sup}_{\mathbf{x} \in K} |u_{n|m}(t, \mathbf{x}) - u^{(m)}(t, \mathbf{x})| \to 0 \qquad \text{as} \quad n \to \infty$$

(ii) $u^{(1)}(t, x)$ depends only on f (not on the particular choice of $\{f_n\}$), is continuous in (t, x), but does not solve the Boltzmann equation (1.1′) unless f satisfies $f(q - tv, v)\, f(q + tv, -v) = f(q - tiv, iv)\, f(q + tiv, -iv)$ for all $x = (q, v)$ and $0 \leqslant t \leqslant t_0$.

Remark 1. The identity mentioned last in Theorem 1 holds if and only if $U_1^0(t) f$ solves (1.1′) (see ref. 12 for an explicit characterization). If this is the case, every term except the first in the series expansion (2.7) vanishes in the Grad limit so that $u^{(1)}(t)$ agrees with $U_1^0(t) f$ and thus solves (1.1′).

The proof of the last claim in Theorem 1, the main assertion of this paper, will be carried out by demonstrating the relation

$$\begin{aligned} &\lim_{t \downarrow 0} t^{-3}[u(t, x) - u^{(1)}(t, x)] \\ &\quad = (32/9)[f(q, iv)^2 f(q, -iv)^2 - f(q, v)^2 f(q, -v)^2] \end{aligned} \tag{2.9}$$

where $x = (q, v)$ and $u(t)$ is a unique solution of the Boltzmann equation (1.1′) with $a = 4$.

The next theorem states that $u^{(m)}(t)$ in Theorem 1 is factorized except on a hypersurface of Ω_m^0. Let $m = m' + m''$ (with $m', m'' \geqslant 1$). Let $\Delta(t; m', m'')$ be the set of all configurations $\mathbf{x} = (\mathbf{x}'; \mathbf{x}'') \in \Omega_m^0$ with $\mathbf{x}' \in \Omega_{m'}^0$

and $\mathbf{x}'' \in \Omega^0_{m''}$ such that some two particles of $\mathbf{x}$, one from $\mathbf{x}'$ and the other from $\mathbf{x}''$, are on a common line at or within the distance $2t$ from each other and their velocities point along the line; in other words, there exists at least one pair (j, k) that satisfies

$$\begin{cases} 1 \leqslant j \leqslant m', & m'+1 \leqslant k \leqslant m \\ v_j \cdot v_k \neq 0, & (q_j - q_k) \cdot (iv_j) = 0, \qquad |q_j - q_k| \leqslant 2t \end{cases} \tag{2.10}$$

Then we have the following result:

Theorem 2. Let $\Delta(t; m', m'')$ be as above. If $\mathbf{x} = (\mathbf{x}'; \mathbf{x}'') \in \Omega^0_m \setminus \Delta(t; m', m'')$, then

$$u^{(m)}(t, \mathbf{x}) = u^{(m')}(t, \mathbf{x}')\, u^{(m'')}(\mathbf{x}'') \tag{2.11}$$

Otherwise $u^{(m)}(t, \mathbf{x})$ is not generally factorized.

Remark 2. In view of (1.3), it must be the behavior of $u_{n|2}(t, q, v; q + l/n, -v)$ that should actually be at issue. Let $\tilde{u}_{n|m}(t)$ denote the right-hand side of (2.7). Then, under the hypothesis of Theorem 1, $\tilde{u}_{n|2}(t, q, v; q + l/n, -v)$ converges uniformly in $(t, q, l) \in K_{\delta, N} := [\delta, t_0] \times [-N, N]^2 \times \partial A$ for every positive δ and N, implying that if we denote the limit by $w(t, l; x)$, $x = (q, v)$, then, as $n \to \infty$,

$$\operatorname*{ess\,sup}_{K_{\delta, N}} |w(t, l; x) - u_{n|2}(t, q, v; q + l/n, -v)| \to 0$$

By passing to the limit in (1.3) we accordingly obtain an equation that is usually expected to agree with the Boltzmann equation (1.1′), but not for the present model: the function $w(t)$ is not resolved into the product $u^{(1)}(t, q, v)\, u^{(1)}(t, q, -v)$.

One might ask what equation characterizes the limit $u^{(1)}(t)$ (it, if any, cannot be local in time). Unfortunately, there seems to exist no such equation that can be neatly and explicitly written down.

2.4. Intuitive Reasoning

It was emphatically pointed out by Grad[4] that the chaos property of the correlation functions is immediately destroyed for configurations in which two particles have just collided and, in order to assure the validity of the Boltzmann equation, it is needed only for those configurations in which two particles are about to collide. With these comments in mind, let us inspect the condition (2.10), which implies that the jth and the kth particles move along a line in common. Then one may agree that (2.11) cannot be expected if a pair (j, k) satisfies (2.10) with $v_j \cdot v_k = -1$ and $(q_j - q_k) \cdot v_j > 0$, i.e., the jth and kth particles must have collided with each other in the past up to $-t$. What seems strange in the conclusion of Theorem 2 is that (2.11) can fail to hold even in the remaining case of (2.10). This can be understood without resorting to the series expansion (2.7), as is discussed below.

Let f_n be the probability density, so that the exposition may be made in terms of probability. The side-to-side collisions may be excluded from our account. First consider the subcase of (2.10) such that $v_j \cdot v_k = 1$, i.e., one runs ahead of the other. We may assume that the jth particle moves ahead of the kth one (see Fig. 3). Then, taking the existence of $u^{(m)}(t) = \lim \tilde{u}_{n|m}(t)$ as granted, we compare $u^{(2)}(t, x_j; x_k)$ with $u^{(1)}(t, x_j)\, u^{(1)}(t, x_k)$ when t is small.

We first compute the one-particle density function $u^{(1)}(t, x_j)$ up to $O(t^2)$. Suppose that the present time is t. There are two ways for the particle to come into its present phase $x_j = (q_j, v_j)$: (A) without any collision and (B) as a result of a (head-on) collision. To avoid inessential details, $f = f(x)$ is treated as if a function of v only. Then the contribution to $u^{(1)}(t, x_j)$ from case A may be written $f(v_j)[1 - 4tf(-v_j)] + O(t^2)$. For case B we consider the probability that the particle j has made a head-on collision during $(0, t)$ and presently occupies a position in a small square centered at q_j with sides of length δ and parallel to coordinate axes. For the sake of simplicity we put $v_j = (1, 0)$ and $q_j = (t, 0)$. Let q^1 and q^2 be the first and second coordinates of $q \in R^2$. Since the probability of making collisions twice is $O(t^2)$, at time zero the particle j must be found either in a strip bounded by four lines $q^2 = q^1 \pm \delta$, $q^1 = 0$, and $q^2 = t$, or else in its reflection with respect to the q^1 axis. Let the particle be at $(q^1, q^1 + h)$, $|h| < \delta$, $0 < q^1 < t$ at time zero. If the collision is at the site (q^1, y), then $|y| < \delta/2$ and $|y - h| < \delta/2$; hence, the region in which the partner particle of the collision should be found at time zero has the area $2\varepsilon(\delta - |h|)$. Thus, the probability in question is, by writting $v = v_j$,

$$4\varepsilon(n-1)t \int_{-\delta}^{\delta} (\delta - |h|)\, dh\, f(iv)\, f(-iv) + O(t^2)$$

The value of the above integral is δ^2 and we consequently see that

$$u^{(1)}(t, x) = f(v)[1 - 4tf(-v)] + 4tf(iv)\, f(-iv) + O(t^2) \qquad (2.12)$$

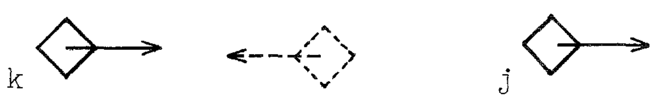

Fig. 3

Now let us turn to the two-particle correlation function $u^{(2)}(t, x_j; x_k)$. If we assume

$$|q_j - q_k| \ll t \tag{2.13}$$

then the probability of the jth particle being subjected to the case B is $O(t^2)$, because if we have case B for the jth particle but not for the kth, then except on a set of (conditional) probability at most $O(t)$ the kth particle must be scattered by the partner particle of the collision by the jth. With the help of (2.12) this shows that under (2.13)

$$u^{(2)}(t, x_j; x_k) = [1 - 4tf(-v)]\, f(v)^2 + 4tf(iv)\, f(-iv)\, f(v) + O(t^2)$$

or, what amounts to the same in view of (2.12),

$$\begin{aligned} &u^{(1)}(t, x_j)\, u^{(1)}(t, x_k) - u^{(2)}(t, x_j; x_k) \\ &\qquad = 4tf(v)[f(iv)\, f(-iv) - f(v)\, f(-v)] + O(t^2) \end{aligned} \tag{2.14}$$

[The last relation is immediate if we make use of (3.1).] Therefore, *two particles that* (like particles j and k in Fig. 2) *are moving on a common line with a common velocity at time t are negatively or positively correlated according as $f(iv)\, f(-iv) - f(v)\, f(-v)$ is positive or negative*, which, as is easily seen, may be valid under $|q_j - q_k| < 2t$ [instead of (2.13)] with t appropriately small.

The conclusion for the case $v_k = v_j$ reached above may explain the other subcase of (2.10) in question, in which two particles are going to collide with each other. Indeed, it suffices to consider another particle moving between them with velocity v_j or v_k. Let it be v_j. Then having such a particle is correlated with the existence of the jth particle and disturbs the course of the kth particle to the effect that this time the jth and the kth particles are *positively or negatively correlated according as $f(iv)\, f(-iv) - f(v)\, f(-v)$ is positive or negative.* [We can directly proceed as before to obtain a concrete expression corresponding to (2.14), though the computation is more complicated: this time the t^2 term becomes relevant.] Now it is rather reasonable to expect the invalidity of the Boltzmann equation for our model according to the comment by Grad, since two particles that are going to collide with each other are correlated even in the Boltzmann–Grad limit.

3. THE PROOF OF THEOREM 1

The proof of Theorem 1 is based on an expression for the right-hand side of (2.7), which will be given in (3.1). First let us introduce some notations. For $\mathbf{x} \in \Omega_m^{(\varepsilon)}$, $l \in \partial\Lambda$, $v \in S$, and $j = 1, 2,..., m$, set

$$C_{j,0}^{v,l}\mathbf{x} = (x_1;...; x_{j-1}; q_j, v_j^*; x_{j+1};...; x_m; q_j - \varepsilon l, v^*)$$
$$C_{j,1}^{v,l}\mathbf{x} = (x_1;...;x_m; q_j + \varepsilon l, v)$$

if $v \cdot l < 0$ and $v_j \cdot l > 0$;

$$C_{j,0}^{v,l}\mathbf{x} = C_{j,1}^{v,l}\mathbf{x} = \partial$$

if $v \cdot l \geqslant 0$ or $v_j \cdot l \leqslant 0$; and

$$C_{j,\sigma}^{v,l}\partial = \partial \qquad (\sigma = 0, 1)$$

(∂ is an extra point). Then

$$K_{m,m+1}\, g(x_1;...; x_m) = 2^{1/2} \sum_{j=1}^{m} \int_{\partial\Lambda \times S} [g(C_{j,0}^{v,l}\mathbf{x}) - g(C_{j,1}^{v,l}\mathbf{x})]\, dl\, dv$$

[Points outside $\Omega_{m+1}^{(\varepsilon)}$ are identified with ∂ and $g(\partial) = 0$ by our convention.] For $\varepsilon \geqslant 0$, $k = 1, 2,..., n-m$, $\mathbf{x} \in \Omega_m^{(\varepsilon)}$, and a set of multivariables

$$\Delta = (\mathbf{s}, \boldsymbol{l}, \mathbf{v}, \boldsymbol{\sigma}, \mathbf{j})$$

where

$$\mathbf{s} = (s_1,..., s_k) \in [0, \infty)^k \qquad \text{with} \quad s_1 < s_2 < \cdots < s_k$$
$$\boldsymbol{l} = (l_1,..., l_k) \in (\partial\Lambda)^k$$
$$\mathbf{v} = (v_{m+1},..., v_{m+k}) \in S^k$$
$$\boldsymbol{\sigma} = (\sigma_1,..., \sigma_k) \in \{0, 1\}^k$$
$$\mathbf{j} = (j_1,..., j_k) \qquad \text{with} \quad 1 \leqslant j_p \leqslant m + p - 1 \quad (p = 1,..., k)$$

we set $M_{0,\Delta}^{(\varepsilon)}\mathbf{x} = \mathbf{x}$ and

$$M_{k,\Delta}^{(\varepsilon)}\mathbf{x} = C_{j_k,\sigma_k}^{v_{m+k}, l_k}\, T_{-s_k + s_{k-1}}^{(m+k)} \cdots C_{j_1,\sigma_1}^{v_{m+1}, l_1}\, T_{-s_1 + s_0}^{(m)}\mathbf{x}$$

where $s_0 = 0$. By writing $|\boldsymbol{\sigma}| = \sum_j \sigma_j$, we can write the right-hand side of (2.7), denoted by $\tilde{u}_{n|m}$, as

$$\tilde{u}_{n|m}(t, \mathbf{x})$$
$$= f_{n|m}(T_{-t}\mathbf{x}) + \sum_{k=1}^{n-m} \sum_{\boldsymbol{\sigma}} \sum_{j_k=1}^{m+k+1} \cdots \sum_{j_1=1}^{m} (-1)^{|\boldsymbol{\sigma}|}(n-m)_k \varepsilon^k 2^{k/2}$$
$$\times \int_0^t ds_k \int_0^{s_k} \cdots \int_0^{s_2} ds_1 \int_{(\partial\Lambda)^k \times S^k} f_{n|m+k}(T_{-t+s_k} M_{k,\Delta}^{(\varepsilon)}\mathbf{x})\, d\boldsymbol{l}\, d\mathbf{v} \qquad (3.1)$$

The mapping $\mathbf{x} \to M_{k,\Delta}^{(\varepsilon)}\mathbf{x}$ is made up of successive applications of the operation of the time-reversed flow $T_{-t}^{(\varepsilon)}$ and the operation $C_{j,\sigma}^{l,v}$ of adding a new particle beside the jth particle. Correspondingly, we shall be concerned with the particle history of an evolving system during a time interval $[0, t]$, determined by t, Δ, and $\mathbf{x} \in \Omega_m^{(\varepsilon)}$, which starts at $\mathbf{x}$ at time zero and ends in $T_{-t+s_k}^{(\varepsilon)} M_{k,\Delta}^{(\varepsilon)}\mathbf{x}$ at time t. In this system new particles are added according to $C_{j,\sigma}^{l,v}$ at times $s_1,..., s_k$, so that the number of particles increases and the system evolves by the time-reversed flow $T_{-s}^{(1/n)}$ during time intervals $[s_j, s_{j+1}]$, $j = 0,..., k$ $(s_0 = 0,\ s_{k+1} = t)$. Let us denote the history of such a system by $\mathbb{F}_{k,\Delta}^{(\varepsilon)}[\mathbf{x}, [0, t]]$.

Proof of (2.9). First we prove (2.9) taking the existence of $\lim \tilde{u}_{n|1}(t) = u^{(1)}(t)$ as granted. Both $u_{n|1}(t)$ and $u(t, x)$ are expressed in series as in (2.7) with $\varepsilon = 1/n$ and as in (2.8), respectively. Let $a_{n|k}(t, x)$ and $a_k(t, x)$ be the kth terms of the series for $u^{(1)}(t)$ and $u(t)$, respectively $(k = 0, 1, 2,...)$. It is easy to see that $\lim_{n\to\infty} a_{n|k} = a_k$ for $k = 0, 1, 2$ and

$$\sum_{k \geq 4} a_k = O(t^4), \qquad \sup_n \sum_{k=4}^{n} |a_{n|k}| = O(t^4)$$

Therefore it suffices to show that

$$a_3^{(1)} := \lim_{n\to\infty} a_{n|3} \quad \text{exists}$$

and $t^{-3}[a_3(t) - a_3^{(1)}(t)]$ converges to the right-hand side of (2.9). To this end, we first observe that $a_{n|3}(t, x)$ is the sum of integrals such as

$$J^n(\mathbf{v}, \boldsymbol{\sigma}, \mathbf{j}) := 2^{3/2}(-1)^{|\boldsymbol{\sigma}|} \int_0^t ds_3 \int_0^{s_3} ds_2 \int_0^{s_2} ds_1 \int_{(\partial\Lambda)^3} f_{n|4}(T_{-t+s_3}^{(1/n)} M_{3,\Delta}^{(1/n)} x)\, d\mathbf{l}$$

over $\mathbf{v}$, $\boldsymbol{\sigma}$, and $\mathbf{j}$, where $\Delta = (\mathbf{s}, \mathbf{l}, \mathbf{v}, \boldsymbol{\sigma}, \mathbf{j})$. The $a_3(t, x)$ is also the sum of similar integrals with $f^{4\otimes}$, $T_{\cdot}^{0}$, and $M_{3,\Delta}^{0}$ in place of $f_{n|4}$, $T_{\cdot}^{(1/n)}$, and $M_{3,\Delta}^{(1/n)}$, which we denote by $J^0(\mathbf{v}, \boldsymbol{\sigma}, \mathbf{j})$. Then

$$\lim_{n\to\infty} J^n(\mathbf{v}, \boldsymbol{\sigma}, \mathbf{j}) = J^0(\mathbf{v}, \boldsymbol{\sigma}, \mathbf{j})$$

except for the following four combinations of $\boldsymbol{\sigma}$, $\mathbf{j}$, and $\mathbf{v}$:

$$\boldsymbol{\sigma} = (1, 1, 1), \qquad \mathbf{j} = (1, 1, 2), \qquad \mathbf{v} = (-v, -v, v)$$

$$\boldsymbol{\sigma} = (1, 1, 1), \qquad \mathbf{j} = (1, 2, 1), \qquad \mathbf{v} = (-v, v, -v)$$

$$\boldsymbol{\sigma} = (0, 1, 1), \qquad \mathbf{j} = (1, 1, 2), \qquad \mathbf{v} = (-v, -iv, iv)$$

$$\boldsymbol{\sigma} = (0, 1, 1), \qquad \mathbf{j} = (1, 2, 1), \qquad \mathbf{v} = (-v, iv, -iv)$$

where v is the velocity component of x. Now we look at the particle history $\mathbb{F}^{(1/n)}_{3,\Delta}[x, [0, t]]$ for each of these four combinations of $\boldsymbol{\sigma}$, $\mathbf{j}$, and $\mathbf{v}$. When the last (i.e., fourth) particle is added at the time s_3, the distance between the fourth and the third particles is at least $2(s_2 - s_1) - 2/n$, so that they can possibly collide with each other by t only if

$$s_2 - s_1 < t - s_3 + 2/(2n) \tag{3.2}$$

Therefore, the limit, as $n \to \infty$, of the integrand for the integral $J^n(\mathbf{v}, \boldsymbol{\sigma}, \mathbf{j})$ possibly differs from the integrand for $J^0(\mathbf{v}, \boldsymbol{\sigma}, \mathbf{j})$ only on the region described by (3.2). But the contribution of the integral over this region to $J^n(\mathbf{v}, \boldsymbol{\sigma}, \mathbf{j})$ is reduced to

$$(-1)^{|\sigma|} 2^{3/2} cAf(q, v) f(q, -v) f(q, iv) f(q, -iv)\, t^3[1 + o(1)]$$

in the limit of having n approach infinity, so that they cancel out one another. Here

$$c := \frac{1}{t^3} \int_0^t ds_3 \int_0^{s_3} ds_2 \int_0^{s_2} I(s_2 - s_1 > t - s_3)\, ds_1 = \frac{1}{24}$$

$$A := 2^{3/2} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} I(-1 < h_1 + h_2 - h_3 < 1)\, dh_1\, dh_2\, dh_3 = \frac{32}{3} \sqrt{2}$$

On the other hand, the corresponding parts for a_3 does not cancel. Consequently, there exists $\lim a_{n|3} =: a^{(1)}$ and

$$\lim_{t \downarrow 0} \frac{1}{t^3} [a_3(t, x) - a_3^{(1)}(t, x)]$$
$$= \frac{32}{9} [f(q, iv)^2 f(q, -iv)^2 - f(q, v)^2 f(q, -v)^2] \quad \blacksquare$$

Proof of (i). Let $k = 1, 2, \ldots$ and $\Delta = (\mathbf{s}, \boldsymbol{l}, \mathbf{v}, \boldsymbol{\sigma}, \mathbf{j})$, where $s = (s_1, \ldots, s_k)$, etc. Set

$$\Gamma_k(t) = \{\mathbf{s} = (s_1, \ldots, s_k) \colon 0 < s_1 < \cdots < s_k < t\}$$

We shall prove that for each $t > 0$, $\boldsymbol{\sigma}$, $\mathbf{j}$, $\mathbf{v}$, and x

$$\mathbf{x}^{(k)}(t, \Delta, x)$$
$$:= \lim_{\varepsilon \to 0} T^{(\varepsilon)}_{-t+s_k} M^{(\varepsilon)}_{k,\Delta} x \quad \text{exists for a.a. } (\mathbf{s}, \boldsymbol{l}) \in \Gamma_k(t) \times (\partial \Lambda)^k \tag{3.3}$$

For the proof of (3.3) we introduce a definition. Given $0 < s < t$, we would like to specify a group of particles in $\mathbb{F}^{(\varepsilon)}_{k,\Delta}[x, [0, t]]$ that run nearly along a line common to them at time s not accidentally but inevitably. We call them *linearly related* at time s. To be precise, this is defined by requiring the following conditions:

(i) Two particles α and β are linearly related at time s if β is born at some time $<s$ beside α by an application of $C^{l,v}_{j,\sigma}$, and their velocities point in directions opposite to each other at β's birth and remain unchanged to time s.

(ii) Particles α and β are linearly related at time s also if at some time $s_1 < s$ the particle β is born beside or makes a side-to-side (not head-on) collision with another particle, say γ, that "has been" linearly related to α up to s_1, being given (or obtaining) a velocity equal or opposite to that which "was" γ's just before time s_1 and the velocities of α and β remain unchanged during the interval $[s_1, s]$.

Given $\mathbf{s}$, $\boldsymbol{\sigma}$, $\mathbf{v}$, $\mathbf{j}$ $[s = (s_1,..., s_k)$, etc., $k \geqslant 3]$, t_0, and x we consider all possible limits or orbits of particles involved in $\mathbb{F}^{(\varepsilon)} := \mathbb{F}^{(\varepsilon)}_{k,\Delta}[x, [0, t_0]]$ as $\varepsilon \downarrow 0$ for a variety of $\boldsymbol{l}$. Except for those that stop at ∂, they may be drawn up by point particles in the corresponding addition-backwardflow evolution, where the head-on collision of two particles gives birth to a pair of particles that take the same courses as two particles do after the collision in $\mathbb{F}^{(\varepsilon)}$, letting the parent particles pass through each other. This system of point particles is denoted by $\mathbb{F}^*$. In $\mathbb{F}^{(\varepsilon)}$ there are head-on collisions that "inevitably" take place for general choices of Δ as considered in the proof of (2.9). To be exact, these are collisions between two particles linearly related at time s and result in their changing velocities for the first time after s. We may "accidentally" have a head-on collision caused by a special choice of $\mathbf{s}$ (this is a head-on collision other than inevitable ones). By arguing using $\mathbb{F}^*$, it is easily seen that, except for $\mathbf{s}$ from a Lebesgue null set, we have:

(a) There is no "accidental" head-on collision in $\mathbb{F}^{(\varepsilon)}$ for sufficiently small ε.

Similarly, except for negligible $\mathbf{s}$, in $\mathbb{F}^*$ there is no twin or multiple collision and every collision time is different form s_i, so that for sufficiently small ε we have:

(b) Every but one pair of two particles in $\mathbb{F}^{(\varepsilon)}$ are apart from each other more than $k\varepsilon$ at each of s_i and of collision times.

In the following discussion we assume that (a) and (b) are valid. It is noted that no side-to-side collision can be "accidental."

Let two particles from $\mathbb{F}^{(\varepsilon)}$, say α and β, be located at time s in such a way that they can make a side-to-side collision between themselves after at most one change of velocity each if their courses to the collision are not intercepted; then we say that α and β are *collaterally related* at time s. Let $x'_t=(q'_t, v'_t)$ and $x''_t=(q''_t, v''_t)$ be the phases of α and β, respectively. For convenience of exposition let them be situated at time s as in Fig. 4 (recall that the motion is backward in time) and suppose that α turns to the left at time $t'>s$ and β to the right at time $t''\in(s,t')$. Put $v=v'_s$. Then $v''_s=iv$ and we see that $a:=(\sqrt{2}\varepsilon)^{-1}(v+iv)\cdot(q''_t-q'_t)$ is independent of $t>s$ before the collision that takes place if and only if $|a|<1/\sqrt{2}$. Suppose $|a|<1/\sqrt{2}$ and the collision occurs at time τ. Then $q''_\tau=q'_\tau+\varepsilon l^{\alpha,\beta}$ where

$$l^{\alpha,\beta}=\frac{1}{\sqrt{2}}a(v+iv)+\frac{1}{2}(v-iv)$$

If α and β repeat a similar evolution after τ, then the same vector $l^{\alpha,\beta}$ relates the position of β to that of α at the time of the next collision. If at time t'' the particle β, instead of turning to the right, collides with or gives birth to another particle, say γ, whose phase just after the collision or its birth is $(q''_{t''}+\varepsilon l_1, v)$ with some $l_1\in\partial\Lambda$, then the vector $l^{\alpha,\gamma}$, which is analogously defined, equals $2^{-1/2}a_1(iv+v)+2^{-1}(v-iv)$, where $a_1=a+2^{-1/2}(v+iv)\cdot l_1$. The other possibility is treated in the same way. The case when α and β are linearly related is discussed analogously (and more simply); in particular, the vector $l^{\alpha,\beta}$ is defined also in such a case. Now it is clear that if the vector $l^{\alpha,\beta}$ is independent of ε for each pair of particles α and β that are linearly or collaterally related at time s, then it is so just after the first collision subsequent to s and hecne ever since s. Thus, the courses taken by particles in $\mathbb{F}^{(\varepsilon)}$ eventually become independent of ε as $\varepsilon\downarrow 0$. This proves in particular the claim (3.3).

The same argument that proved (3.3) (see also the beginning of the proof of Theorem 2 in Section 4) proves that if $\mathbf{x}\in\Omega^0_m$ (note that the case $\mathbf{x}\in d_m$ is trivial), then for a.a. $(\mathbf{s},\mathbf{l})\in\Gamma_k(t)\times(\partial\Lambda)^k$

$$T^{(\varepsilon)}_{-t+s_k}M^{(\varepsilon)}_{k,\Delta}\mathbf{x}\quad\text{converges as}\quad\varepsilon\downarrow 0$$

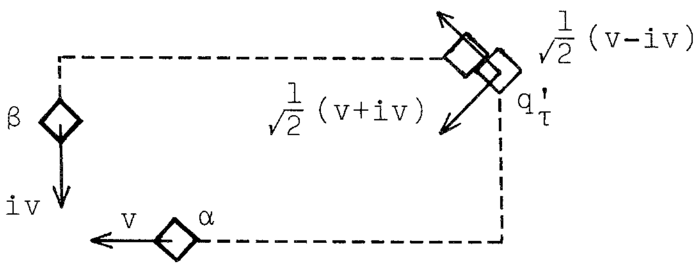

Fig. 4. Two particles α and β are located at time s in such a way that they can make a side-to-side collision between themselves after just one change of velocity each in their motion backward in time.

Hence $\tilde{u}_{n|m}(t, \mathbf{x})$ converges in view of the expression (3.1). The claimed uniformity of the convergence is proved as in the classical case (ref. 11, Appendix II). ▌

Proof of (ii). We have already proved (2.9). The continuity of $u^{(1)}(t, x)$ follows from (3.3) in the proof of (i). In fact, if we set

$$g(t, x, \mathbf{s}) = \int_{(\partial\Lambda)^k} f^{(k+1)\otimes}(\mathbf{x}^{(k)}(t, \Delta, x))\, dl$$

[where $\mathbf{x}^{(k)}$ is defined in (3.3)] with k, $\mathbf{v}$, $\boldsymbol{\sigma}$, and $\mathbf{j}$ fixed, then g is continuous in x for each t, $\mathbf{s}$, because any shift of q $[x = (q, v)]$ only results in the same shift of each position component of $\mathbf{x}^{(k)}(t, \Delta, x)$. And, as $h \to 0$,

$$\begin{aligned}
&\int_0^{t+h} ds_k \int_0^{s_k} \cdots \int_0^{s_2} ds_1\, g(t+h, x, \mathbf{s}) \\
&\quad = \int_h^{t+k} ds_k \cdots \int_h^{s_2} ds_1\, g(t+h, x, \mathbf{s}) + O(h) \\
&\quad = \int_0^{t} ds_k \cdots \int_0^{s_2} ds_1\, g(t, T^0_{-h}x, \mathbf{s}) + O(h)
\end{aligned}$$

where we applied the relation

$$g(t, x, s_1 + h, \ldots, s_k + h) = g(t - h, T^0_{-h}x, s_1, \ldots, s_k)$$

This shows that $u^{(1)}(t+h, x) = u^{(1)}(t, T^0_{-h}x) + O(h)$, because $u^{(1)}$ is a uniformly convergent series of such integrals of $g(t, x, \mathbf{s})$ as above. Therefore $u^{(1)}(t, x)$ is continuous in (t, x). ▌

4. THE PROOF OF THEOREM 2

Before proceeding to the proof of Theorem 2, we introduce some definitions, which will allow us to write down Eq. (4.4) below. For $m = 1, 2, \ldots$, $k = 1, 2, \ldots$, the symbol $J_{m,k}$ denotes a set of multi-indices $\mathbf{j} = (j_1, \ldots, j_k)$ such that

$$1 \leqslant j_1 \leqslant m, \qquad 1 \leqslant j_2 \leqslant m+1, \ldots, \qquad 1 \leqslant j_k \leqslant m+k-1$$

Let $m' \geqslant 1$ and $m'' \geqslant 1$ be such that $m' + m'' = m$. We define $\mathbf{j}' \in J_{m',k'}$ and $\mathbf{j}'' \in J_{m'',k''}$, which are to be determined by $\mathbf{j} \in J_{m,k}$ together with m' and m'', in such a way that in the particle addition scheme specified by $\mathbf{j}$ and $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$, $\mathbf{x}' \in \Omega_{m'}$, $\mathbf{x}'' \in \Omega_{m''}$, the decomposition of $\mathbf{j}$ into $\mathbf{j}'$ and $\mathbf{j}''$ corresponds to the (natural) decomposition of the "descendents" of $\mathbf{x}$ into those of $\mathbf{x}'$ and of $\mathbf{x}''$. To do this we introduce a family tree of integers associated with $\mathbf{j}$. There are $k+1$ generations in the tree. The νth generation consists of integers $1, 2,..., m+\nu$ and is decomposed into two families I'_ν and I''_ν, $\nu = 0, 1,..., k$, which are inductively defined by the following conditions:

$$I'_0 = \{1,..., m'\}, \qquad I''_0 = \{m'+1,..., m\}$$

and for $\nu \geqslant 1$

$$I'_\nu = I'_{\nu-1} \cup \{m+\nu\} \quad \text{and} \quad I''_\nu = I''_{\nu-1} \qquad \text{if} \quad j_\nu \in I'_{\nu-1}$$
$$I'_\nu = I'_{\nu-1} \quad \text{and} \quad I''_\nu = I''_\nu \cup \{m+\nu\} \qquad \text{if} \quad j_\nu \in I''_{\nu-1}$$

If j_ν gives birth to the $(m'+r)$th member of I'_ν, i.e.,

$$j_\nu \in I'_{\nu-1} \qquad \text{and} \qquad r = \# I'_\nu - m' \tag{4.1}$$

($\#$ denotes the cardinality of a set) and if j_ν equals the jth element of $I'_{\nu-1}$ (elements of I'_ν and I''_ν are arranged in increasing order), then we set $j'_r = j$ and define $k' = \# I'_k - m'$ and

$$\begin{aligned} \mathbf{j}' &= (j'_1,..., j'_{k'}) \qquad &\text{if} \quad k' > 0 \\ &= 0 \qquad &\text{if} \quad k' = 0 \end{aligned} \tag{4.2}$$

$\mathbf{j}''$ is defined similarly. Let $\mathbf{s} = (s_1,..., s_k)$, etc., be given as before. According to the transform $\mathbf{j} \to (\mathbf{j}', \mathbf{j}'')$, we define the decompositions $(\mathbf{s}', \mathbf{s}'')$, $(\boldsymbol{l}', \boldsymbol{l}'')$, $(\boldsymbol{\sigma}', \boldsymbol{\sigma}'')$, and $(\mathbf{v}', \mathbf{v}'')$: for example,

$$s'_r = s_\nu \tag{4.3}$$

if (4.1) holds (s_ν is attached to the first family or the second according as j_ν belongs to I'_ν or I''_ν).

Proof of Theorem 2. By the same argument as in the proof of (3.3) we can show that if $\mathbf{x}'$ and $\mathbf{x}''$ satisfy the hypothesis of Theorem 2, then for a.a. $(\mathbf{s}, \boldsymbol{l}) \in \Gamma_k(t) \times (\partial \Lambda)^k$ there occurs no collision between any descendant of $\mathbf{x}'$ and any one of $\mathbf{x}''$ in $\mathbb{F}^{(\varepsilon)}_{k,\Delta}[\mathbf{x}, [0, t]]$ eventually as $\varepsilon \downarrow 0$. This, together with the proof of (i) of Theorem 1, implies that for a.a. $(\mathbf{s}, \boldsymbol{l})$ (and all $\mathbf{v}$, $\boldsymbol{\sigma}$, $\mathbf{j}$)

$$\begin{aligned} &\lim_{n\to\infty} f_{n|k+m}(T^{(1/n)}_{-t+s_k} M^{(1/n)}_{k,\Delta}(\mathbf{x}', \mathbf{x}'')) \\ &\quad = f^{(m'+k')\otimes}\left(\lim_{n\to\infty} T^{(1/n)}_{-t+s'_{k'}} M^{(1/n)}_{k',\Delta'}\mathbf{x}'\right) \\ &\qquad \times f^{(m''+k'')\otimes}\left(\lim_{n\to\infty} T^{(1/n)}_{-t+s''_{k''}} M^{(1/n)}_{k'',\Delta''}\mathbf{x}''\right) \end{aligned} \tag{4.4}$$

where $\Delta^i = (\mathbf{s}^i, \boldsymbol{l}^i, \mathbf{v}^i, \boldsymbol{\sigma}^i, \mathbf{j}^i)$, $i = {}'$ or ${}''$.

Let us fix t, $\mathbf{x}'$, and $\mathbf{x}''$ which together satisfy the hypothesis of Theorem 2, and set, for $s \in \Gamma_k(t)$ and $\mathbf{j} \in J_{m,k}$

$$f'_0 = f(T^0_{-t}\mathbf{x}')$$

$$f'_{j'}(\mathbf{s}') = f'_{j'}(s'_1,\dots, s'_{k'})$$

$$= \sum_{\sigma'} (-1)^{|\sigma'|} \iint f^{(m'+k')\otimes}(\lim T^{1/n}_{-t+s'_{k'}} M^{(1/n)}_{k',\Delta'}\mathbf{x}')\, d\mathbf{l}'\, d\mathbf{v}'$$

$f''_{\mathbf{j}''}(\mathbf{s}'')$ is defined similarly. (Below, statements like this will be omitted on the understanding that the definition for an object carrying double primes is parallel.) Then

$$\lim_{n\to\infty} u_{n|m}(t, \mathbf{x})$$

$$= f'_0 f''_0 + \sum_{k=1}^{\infty} \sum_{\mathbf{j}\in J_{m,k}} \int_0^t ds_k \int_0^{s_k} ds_{k-1} \cdots \int_0^{s_2} ds_1 f'_{\mathbf{j}'}(\mathbf{s}') f''_{\mathbf{j}''}(\mathbf{s}'')$$

The required relation $u^{(m)}(t, \mathbf{x}) = u^{(m')}(t, \mathbf{x}')\, u^{(m'')}(t, \mathbf{x}'')$ follows from the next lemma.

Lemma. Let $f'_{\mathbf{j},m}(\mathbf{s})$, $f''_{\mathbf{j},m}(\mathbf{s})$, $\mathbf{j} \in J_{m,k}$ be a family of bounded, measurable functions of $\mathbf{s}$: $0 < s_1 < \cdots < s_k < t$ ($k = 1, 2,\dots$, $m = 1, 2,\dots$) and $f'_{0,m}$ and $f''_{0,m}$ real numbers ($m = 1, 2,\dots$). Then, for every k, m, and pair (m', m'') with $m' + m'' = m$,

$$\sum_{\mathbf{j}\in J_{m,k}} \int_0^t ds_k \int_0^{s_k} ds_{k-1} \cdots \int_0^{s_2} ds_1\, f'_{\mathbf{j}',m'}(\mathbf{s}') f''_{\mathbf{j}'',m''}(\mathbf{s}'')$$

$$= \sum_{k'+k''=k} F'_{m',k'} F''_{m'',k''} \tag{4.5}$$

where

$$F'_{m,0} = f'_{0,m}$$

$$F'_{m,k} = \sum_{\mathbf{j}\in J_{m,k}} \int_0^t ds_k \int_0^{s_k} \cdots \int_0^{s_2} ds_1\, f'_{\mathbf{j},m}(\mathbf{s})$$

[similarly for $F''_{m,k}$; $\mathbf{j} \to (\mathbf{j}', \mathbf{j}'')$ and $\mathbf{s} \to (\mathbf{s}', \mathbf{s}'')$ are defined by (4.1)–(4.3)].

Proof. Let us reverse the order of integration on the left-hand side of (4.5). If we set

$$g'_{\mathbf{j},m}(s_1,\dots, s_k) = f'_{\mathbf{j},m}(t - s_1,\dots, t - s_k)$$

then it becomes

$$\sum_{\mathbf{j} \in J_{m,k}} \int_0^t ds_1 \int_0^{s_1} \cdots \int_0^{s_{k-1}} ds_k \; g'_{\mathbf{j}',m'}(\mathbf{s}')\, g''_{\mathbf{j}'',m''}(\mathbf{s}'') \tag{4.6}$$

and also

$$F'_{m,k} = \sum_{\mathbf{j} \in J_{m,k}} \int_0^t ds_1 \cdots \int_0^{s_{k-1}} ds_k \; g'_{\mathbf{j}',m'}(\mathbf{s})$$

We proceed by induction on k. For $k = 1$, (4.5) holds. Let (4.5) be valid for $k = p$ and for all $m = 1, 2,\ldots$. For $\mathbf{j} \in J_{m+1,k}$, $j = 1,\ldots, m$, and $s_1 < s < t$ set

$$\tilde{g}'_{j,\mathbf{j},m}(s, \mathbf{s}) = g'_{(j,\mathbf{j}),m}(s, s_1,\ldots, s_k)$$

where $(j, \mathbf{j}) = (j, j_1,\ldots, j_k) \in J_{m,k+1}$. Then (4.6) with $p+1$ in place of k is written

$$\sum_{j=1}^{m'} \sum_{\mathbf{j} \in J_{m+1,p}} \int_0^t ds \int_0^s ds_1 \cdots \int_0^{s_{p-1}} ds_p \; \tilde{g}'_{j,\mathbf{j}',m'}(s, \mathbf{s}')\, g''_{\mathbf{j}'',m''}(\mathbf{s}'')$$
$$+ \sum_{j=1}^{m''} \sum_{\mathbf{j} \in J_{m+1,p}} \int_0^t ds \int_0^s ds_1 \cdots \int_0^{s_{p-1}} ds_p \; g'_{m',\mathbf{j}'}(\mathbf{s}')\, \tilde{g}''_{j,\mathbf{j}'',m''}(s, \mathbf{s}'')$$

[Recall $\mathbf{s} \to (\mathbf{s}', \mathbf{s}'')$ is determined by $\mathbf{j}$.] By the induction hypothesis this equals

$$\sum_{j=1}^{m'} f''_{0,m''} \int_0^t \tilde{F}'_{m',j,p}(s)\, ds + \sum_{j=1}^{m''} f'_{0,m'} \int_0^t F''_{m'',j,p}(s)\, ds$$
$$+ \sum_{j=1}^{m'} \sum_{k'+k''=p} \int_0^t ds\, \tilde{F}'_{m',j,k'}(s) \int_0^s h''_{m'',k''-1}(s_1)\, ds_1$$
$$+ \sum_{j=1}^{m''} \sum_{k'+k''=p} \int_0^t ds\, \tilde{F}''_{m,j,k''}(s) \int_0^s h'_{m',k'-1}(s_1)\, ds_1 \tag{4.7}$$

where

$$\tilde{F}'_{m,j,0}(s) = \tilde{g}_{j,m}(s)$$

$$\tilde{F}'_{m,j,k}(s) = \sum_{\mathbf{j} \in J_{m+1,k}} \int_0^s ds_1 \cdots \int_0^{s_{k-1}} ds_k \; \tilde{g}'_{j,\mathbf{j},m}(s, \mathbf{s})$$

$$h'_{m,0}(s_1) = \sum_{j=1}^{m} g'_{j,m}(s_1)$$

$$h'_{m,k-1}(s_1) = \sum_{\mathbf{j} \in J_{m,k}} \int_0^{s_1} ds_2 \cdots \int_0^{s_{k-1}} ds_k \; g'_{\mathbf{j},m}(\mathbf{s})$$

By the relations

$$\sum_{j=1}^{m} \tilde{F}^i_{m,j,k}(s) = h^i_{m,k}(s), \qquad \int_0^t h^i_{m,k-1}(s)\, ds = F^i_{m,k}$$

($i = {}'$ and $''$), which are immediate from definitions, we see that (3.9) equals

$$\sum_{k'+k''=p+1} F'_{m',k'} F''_{m'',k''}$$

This completes the proof of the lemma. ∎

APPENDIX

In this Appendix f_n is a symmetric probability density satisfying the continuity condition (2.4). We consider the n-particle system described in Section 2. We show that recollisions are negligible in the Grad limit at least for short times under some boundedness conditions of $f_{n|m}$. Here the recollision is taken in a broad sense: it is a collision between descendant particles of the prescribed progenitor particle (or particles).

Given a particle (the progenitor), a particle is its *descendant* in the time interval $(0, t)$ if (i) it is the progenitor itself, (ii) it collides with the progenitor during $(0, t)$, or (iii) it collides with a descendant of the given particle during (s, t), where s is the time of the collision that makes the latter a descendant in $(0, t)$. (To be precise, the set of descendants is defined as the smallest class that satisfies these three conditions.) A descendant is said to be born at the time s if s is the time of the collision that makes it a descendant. (The birth time of the progenitor is taken to be 0.) We count as *collisions between two descendants* of a common progenitor only those that take place after the time they are born.

Proposition 1. Let $t > 0$ and assume that there exist constants M and C such that $8Mt < 1$ and

$$\|u_{n|m}(t)\|_\infty \leqslant CM^m \qquad \text{for all} \quad m \leqslant n, \quad n = 1, 2,... \tag{A.1}$$

Then the probability density (with respct to dx) for the event that the first particle is initially at x and there is at least one head-on collision between descendants of the first particle in $(0, t)$ vanishes in the Grad limit uniformly in x.

We can prove analogous statements in the case when more than two progenitors are involved. What we are especially interested in is

Proposition 2 below, in which we are concerned with a conditional probability of recollisions, given that

$$\text{the first and second particles constitute the phase}$$
$$\mathbf{x}_2 = (q, v; q + \varepsilon l, -v) \in \partial\Omega_2^{(\varepsilon)} \text{ at time } 0 \tag{A.2}$$

Let $l \cdot v < 0$. Then two particles in (A.2) are in the outgoing collision. The corresponding precollisional phase is denoted by $\mathbf{x}_2^*$. To have the conditional probability well defined, we assume that

$$f_{n|2}(\mathbf{x}_2) = f_{n|2}(\mathbf{x}_2^*);\ f_{n|2}(\cdot) \text{ is continuous at both } \mathbf{x}_2 \text{ and } \mathbf{x}_2^* \tag{A.3}$$

Proposition 2. Given $(q, v) \in \Omega_1^0$, $l \in \partial\Lambda$, with $-1 < l \cdot v < 0$, let $f_{n|2}$ satisfy the regularity condition (A.3). If (A.1) holds and $f_{n|2}(\mathbf{x}_2)$ is bounded away from zero, then the conditional probability, given (A.2), of the event that there is a collision between one of the descendants of the first particle and that of the second in $(0, t)$ vanishes in the Grad limit.

As a dual notion to "descendants," we can define the "ancestors" by means of the time-reversed flow and state a corresponding result by interchanging the roles of time 0 and time t [and accordingly also those of $u_n(t)$ and f_n] in Propositions 1 and 2.

For the proof of the propositions we seek an upper bound of the probability that there are just k descendants of prescribed particles. For $x = (q, v)$, $t > 0$, and $\Delta = (\mathbf{s}, \boldsymbol{l}, \mathbf{v}, \mathbf{j})$ [with $\mathbf{s} = (s_1,..., s_k)$, $s_1 < \cdots < s_k$, $\mathbf{v} = (v_1,..., v_k)$, etc., as in Section 3, but with $\boldsymbol{\sigma}$ omitted] let $p_k(t, x, \Delta)$ [resp. $\tilde{p}_k(t, x, \Delta)$] denote the probability density (with respect to $dx\, d\mathbf{s}\, d\boldsymbol{l}$) for the event that the first particle is initially at x and has just [resp. at least] $k+1$ descendants (including the progenitor itself) in $(0, t)$, the νth, $\nu = 1,..., k$, of which is born with the (precollisional) velocity v_ν at the site deviating by $-\varepsilon l_\nu$ from the site of the j_νth descendant at the time s_ν. Then we have the following result:

Lemma. The probability density $p_k(t, x, \Delta)$ is dominated by

$$\varepsilon^k (n-1)_k 2^{k/2} u_{n|k+1}(t, T_{t-s_k} C_{j_k,1}^{v_k,l_k} T_{s_k - s_{k-1}} \cdots C_{j_1,1}^{v_1,l_1} T_{s_1} x) \tag{A.4}$$

The index σ in $C_{j,\delta}^{v,l}$ appearing above equals one, meaning that the added particle together with the jth particle at first forms an incoming configuration with given velocities, which is to be transformed into the outgoing configuration as soon as T_s operates. [If we are concerned with ancestors instead of descendants, (A.4) is to be replaced by an analogous expression with $\sigma_j = 0$ that appears in the series expansion (3.1).]

Proof of Lemma. Let $k=1$. Then, since the free motion (trivially) preserves the area dq, $\tilde{p}_1(t, x, \Delta)\, dq\, ds_1\, dl_1$ may be written

$$(n-1)\varepsilon 2^{1/2}\, dq\, ds_1\, dl_1 \int^*_{\Omega^{(\varepsilon)}_{n-2}} u_n(s_1, C^{v_1, l_1}_{1,1} T^0_{s_1} x; \mathbf{x}^{(2)})\, d\mathbf{x}^{(2)} \tag{A.5}$$

Here the star indicates that the integral extend over those $\mathbf{x}^{(2)} = (x_3; ...; x_n)$ that do not cause the first particle to undergo any collision during $(0, s_1)$; in other words, in the flow (backward in time) starting with $(C^{v_1, l_1}_{1,1} T^0_{s_1} x; \mathbf{x}^{(2)})$ the first particle makes no collision in the past period $(-s_1, 0)$. By the measure-preserving property of $T^{(2)}_t$ and $T^{(n-2)}_t$ the quantity in (A.5) agrees with

$$(n-1)\varepsilon 2^{1/2}\, dq\, ds_1\, dl_1 \int^*_{\Omega^{(\varepsilon)}_{n-2}} u_n(s_1 + ds_1, T^{(2)}_{ds_1} C^{v_1, l_1}_{1,1} T^0_{s_1} x; \mathbf{x}^{(2)})\, d\mathbf{x}^{(2)} \tag{A.6}$$

up to $o(dq\, ds_1\, dl_1)$, which in turn equals $(n-1)$ times the probability of the event that the first and second particles are in a neighborhood of $T^{(2)}_{ds_1} C^{v_1, l_1}_{1,1} T^0_{s_1} x$ of area $\varepsilon\sqrt{2}\, dq\, ds_1\, dl_1$ and the first particle has experienced no collision during $(0, s_1)$. Therefore, if we further restrict the range of the integral in (A.6) to those $\mathbf{x}^{(2)}$ that do not cause the first and second particles to undergo any collision during $(s_1 + ds_1, t)$, then the resultant expression gives the probability $p_1(t, x, \Delta)\, dq\, ds_1\, dl_1$. Now applying the transformation $T^{(n)}_{t-s_1-ds_1}$ to change the variable of integration and thereafter omitting the all constraints on $\mathbf{x}^{(2)}$, we finally get an upper bound asserted by the lemma. For $k \geqslant 2$, starting from the expression (A.6) of $\tilde{p}_1(t, x, (s_1, ..., j_1))\, dq\, ds_1\, dl_1$, we can proceed as above. ∎

Remark. The proof above also shows

$$\tilde{p}_k(t, x, \Delta) \leqslant \varepsilon^k (n-1)_k 2^{k/2} u_{n|k+1}(s_k, C^{v_k, l_k}_{j_k, 1} T_{s_k - s_{k-1}} \cdots C^{v_1, l_1}_{j_1, 1} T_{s_1} x)$$

Proof of Proposition 1. Because of the lemma and the hypothesis (A.1), it suffices to show that in the Grad limit the probability of the event that the number of descendants of the first particle is $k+1$ vanishes, and there is a collision between two of them before the time t. But this is ready from the lemma, since in every limiting trajectory of $k+1$ descendants except those for a null set of **s** and ***l*** no two (point) particles run straight against each other on a common line. ∎

The proof of Proposition 2 is similarly carried out by extending the lemma to the case of two progenitors.

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